Exercise 17

Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

$$y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + 10y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + 2(re^{rx}) + 10(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 2r + 10 = 0$$

Solve for r.

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

$$r = \{-1 - 3i, -1 + 3i\}$$

Two solutions to the ODE are $e^{(-1-3i)x}$ and $e^{(-1+3i)x}$. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{(-1-3i)x} + C_2 e^{(-1+3i)x} \\ &= C_1 e^{-x} e^{-3ix} + C_2 e^{-x} e^{3ix} \\ &= e^{-x}(C_1 e^{-3ix} + C_2 e^{3ix}) \\ &= e^{-x}[C_1(\cos 3x - i \sin 3x) + C_2(\cos 3x + i \sin 3x)] \\ &= e^{-x}[(C_1 + C_2)\cos 3x + (-iC_1 + iC_2)\sin 3x] \\ &= e^{-x}(C_3 \cos 3x + C_4 \sin 3x). \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 2y_p' + 10y_p = x^2 e^{-x} \cos 3x$$

Since the inhomogeneous term is the product of a polynomial and an exponential and a cosine, the particular solution would be

$$y_p = (Ax^2 + Bx + C)e^{-x}(D \cos 3x + E \sin 3x).$$

$e^{-x} \cos 3x$ already satisfies the complementary solution, though, so an extra factor of x is needed.

$$y_p = x(Ax^2 + Bx + C)e^{-x}(D \cos 3x + E \sin 3x)$$